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NONLINEAR DYNAMIC RESPONSE OF A FLEXIBLE THIN PLATE TO CONSTANT ACCELERATION APPLIED TO ITS SUPPORT CONTOUR, WITH APPLICATION TO PRINTED CIRCUIT BOARDS, USED IN AVIONIC PACKAGING

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Abstract—We evaluate the dynamic response of a flexible rectangular thin plate to a constant acceleration applied to its support contour. The study is carried out in application to printed circuit boards (PCBs) used in avionic packaging and is based on a simple and easy-to-use analytical model, clearly indicating the role of various factors affecting the mechanical behavior of the board. The main goal of the study is to determine the maximum accelerations experienced by the board. We showed that in the case of large deflections of a board with a nondeformable contour it is important to account for nonlinear effects, which are due to the membrane forces and lead to substantially higher accelerations and stresses. The developed theory can be helpful in evaluating the accelerations acting on surface-mounted components, and choosing the appropriate PCB type, dimensions and support conditions. It can also be used when choosing the most feasible layout of the electronic devices on the board.

INTRODUCTION

Although cyclic differential thermal expansion is usually regarded as the most typical and the most critical type of loading on electronic equipment, mechanical, and especially dynamic, loading can play a crucial role in the performance and reliability of electronic components and devices [see, for instance, Steinberg (1973, 1989); Suhir and Lee (1989)]. Such loading can occur during mechanical handling or accidental misuse of the equipment, as well as during its shipment (transportation). In military applications, dynamic loading takes place even during normal operation of the electronic equipment.

In the analysis below we evaluate the dynamic response of a flexible thin plate to a constant acceleration, suddenly applied to its support contour, with application to printed circuit boards (PCBs) used in avionic packaging. Shock loading of this type can occur, for example, during launching or manoeuvering of a spacecraft or a guided missile.

The main goal of our investigation is to determine the maximum accelerations experienced by the electronic components and devices surface-mounted (SM) on the board. It is known that elevated accelerations can affect both the mechanical integrity and the normal functioning of the SM components. In addition, we evaluate the maximum stresses in the PCB in order to establish whether these stresses can be high enough to cause any serious damage to the board structure.

In our study we consider the fact that the PCB's support contour is typically nondeformable. This leads to reactive membrane (in-plane) stresses which are proportional to the deflections squared [see, for instance, Timoshenko and Woinowski-Krieger (1969)]. Limiting our analysis to the principal mode of vibrations, we obtain an exact solution to the nonlinear problem in question. Clearly, such a solution can be applied with confidence even in the case of significant external accelerations and very large elastic deflections. Linear theory, which considers bending only, always overestimates the bending stresses, as well as the dynamic deflections (amplitudes). When these deflections are small, and the vibration frequency can be assumed deflection independent, the linear approach conservatively and quite accurately predicts the induced accelerations and stresses. However, when the deflections are significant, the linear approach may not be conservative. As far as the stresses are concerned, this is due to the membrane forces which are ignored by the linear theory. In the case of the maximum accelerations, this is due to the fact that the nonlinear frequency can be substantially larger than the frequency of linear vibrations, and the induced accelerations are approximately proportional to the frequency of vibrations squared.

In this investigation, we treat a PCB as a thin and flexible rectangular plate. Note that various linear and nonlinear problems of the dynamics of plate structures were analyzed and discussed in great detail in numerous monographs, manuals and reference books. Examples are: Timoshenko and Young (1955), Den-Hartog (1956) and Harris and Crede (1976). Approximate analyses of nonlinear oscillators with stiff cubic characteristics (so-called Duffing oscillators) can be found, for instance, in the books by Kauderer (1958) and Hayashi (1964). It should be pointed out, however, that the emphasis is made in these books and manuals on periodic excitations and weak nonlinearities. Solutions to static problems of nonlinear bending of plates were presented by Bubnov (1912), Prescott (1924), Levy (1942) and Timoshenko and Woinowski-Krieger (1969). Linear modal analyses and dynamic response predictions of PCBs and SM components, subjected to continuous vibrations, have been carried out recently by Lau and Keely (1989), Crovetto *et al.* (1990), Keltie and Ozisik (1990), Kim and Gupta (1990) and Wong *et al.* (1990). Structural analysis of circuit board (card) systems subjected to bending was performed by Engel (1990).

ANALYSIS

Stress function

The membrane stresses σ_x^0, σ_y^0 (normal) and τ_{xy}^0 (shearing) in a PCB are expressed through the stress function ϕ as follows [see, for instance, Timoshenko and Woinowski-Krieger (1969)]:

$$\sigma_x^0 = \frac{\partial^2\phi}{\partial y^2}, \quad \sigma_y^0 = \frac{\partial^2\phi}{\partial x^2}, \quad \tau_{xy}^0 = -\frac{\partial^2\phi}{\partial x\,\partial y}. \tag{1}$$

This function must satisfy the continuity equation

$$\nabla^4\phi = -EL(w, w), \tag{2}$$

where $w = w(x, y, t)$ is the deflection function, E is Young's modulus of the material, and the operators ∇^4 (biharmonic operator) and L are:

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2\,\partial y^2} + \frac{\partial^4}{\partial y^4}, \quad L = \frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial y^2} - \left(\frac{\partial^2}{\partial x\,\partial y}\right)^2.$$

In the case of a PCB with a nondeformable contour, the stress function ϕ must also satisfy the conditions

$$\left.\begin{aligned} \frac{1}{E}\int_0^{a/2} (\sigma_x^0 - \nu\sigma_y^0)\,\mathrm{d}x &= \frac{1}{2}\int_0^{a/2}\left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x \\ \frac{1}{E}\int_0^{b/2} (\sigma_y^0 - \nu\sigma_x^0)\,\mathrm{d}y &= \frac{1}{2}\int_0^{b/2}\left(\frac{\partial w}{\partial y}\right)^2 \mathrm{d}y \end{aligned}\right\}, \tag{3}$$

where ν is Poisson's ratio of the PCB material, and a and b are the PCB dimensions in the

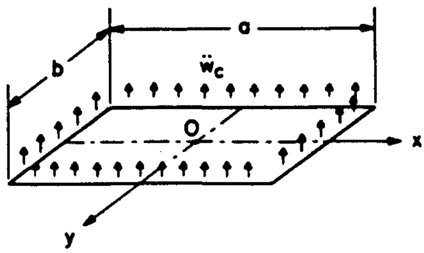

Fig. 1. Flexible thin plate (printed circuit board) under the action of constant acceleration applied to its support contour.

x and y directions, respectively (Fig. 1). Conditions (3) simply state that the in-plane displacements due to the membrane stresses must be equal to the in-plane displacements due to bending. Note that although the elastic constants of a PCB are, generally speaking, somewhat different in the directions x and y, in this study we do not account for this difference, i.e. assume that the PCB material is isotropic. In addition, we assume that this material has the same properties in all its points, i.e. homogeneous.

Limiting our analysis to the first mode of vibrations, we seek the functions w and ϕ in the form:

$$w = w_c(t) - w_1(x,y)z(t), \quad \phi = \phi_1(x,y)z^2(t), \tag{4}$$

where $w_c(t)$ is the displacement of the support contour, $w_1(x,y)$ is the coordinate function of the first mode of vibrations, $z(t)$ is the corresponding principal coordinate, and $\phi_1(x,y)$ is the static stress function.

The coordinate function w_1 of a simply-supported board with a finite aspect ratio b/a between 1 and 2 can be presented, when the membrane stresses are sought, in the form (Prescott, 1924):

$$w_1 = \cos\frac{\pi x}{a}\cos\frac{\pi y}{b}. \tag{5}$$

After substituting (4), with consideration of (5), into the continuity eqn (2) and the non-deformability conditions (3), we find that the function ϕ_1 must satisfy the equation

$$\nabla^4\phi_1 = -\frac{\pi^4 E}{2a^2b^2}\left(\cos\frac{2\pi x}{a} + \cos\frac{2\pi y}{b}\right)$$

and the conditions

$$\frac{1}{E}\int_0^{a/2}\left(\frac{\partial^2\phi_1}{\partial y^2} - \nu\frac{\partial^2\phi_1}{\partial x^2}\right)\mathrm{d}x = \frac{\pi^2 b}{32a}, \quad \frac{1}{E}\int_0^{b/2}\left(\frac{\partial^2\phi_1}{\partial x^2} - \nu\frac{\partial^2\phi_1}{\partial y^2}\right)\mathrm{d}y = \frac{\pi^2 b}{32b}.$$

These result in the following expression for the static stress function:

$$\phi_1 = \frac{E}{32}\left\{\frac{2\pi^2}{1-\nu^2}\left[\left(\frac{\nu}{a^2} + \frac{1}{b^2}\right)x^2 + \left(\frac{1}{a^2} + \frac{\nu}{b^2}\right)y^2\right] - \frac{a^2}{b^2}\cos\frac{\pi x}{a} - \frac{b^2}{a^2}\cos\frac{\pi y}{b}\right\}. \tag{6}$$

The coordinate function for a clamped board (plate) with an aspect ratio b/a between 1 and 1.5 can be assumed in the following approximate form (Levy, 1942):

$$w_1 = \cos^2\frac{\pi x}{a}\cos^2\frac{\pi y}{b}. \tag{7}$$

This, together with the conditions of the nondeformability of the contour, results in the following formula for the static stress function:

$$\phi_1 = \frac{E}{32}\left\{\frac{3\pi^2}{2(1-\nu)^2}\left[\left(\frac{\nu}{a^2}+\frac{1}{b^2}\right)x^2+\left(\frac{1}{a^2}+\frac{\nu}{b^2}\right)y^2\right]-\frac{a^2}{b^2}\cos\frac{\pi x}{a}-\frac{b^2}{a^2}\cos\frac{\pi y}{b}\right.$$
$$-\left(\frac{a}{4b}\right)^2\cos\frac{4\pi x}{a}-\left(\frac{b}{4a}\right)^2\cos\frac{4\pi y}{b}-\frac{2a^2b^2}{(a^2+b^2)^2}\cos\frac{2\pi x}{a}\cos\frac{2\pi y}{b}$$
$$\left.-\frac{a^2b^2}{(4a^2+b^2)^2}\cos\frac{2\pi x}{a}\cos\frac{4\pi y}{b}-\frac{a^2b^2}{(a^2+4b^2)^2}\cos\frac{4\pi x}{a}\cos\frac{2\pi y}{b}\right\}. \tag{8}$$

Equation of motion

The kinetic energy T and the strain energy V of a PCB (plate), subjected to bending and membrane forces, are expressed as follows [see, for instance, Timoshenko and Woinowski-Krieger (1969)]:

$$T = \tfrac{1}{2}m\int_A\left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}A, \tag{9}$$

$$V = \tfrac{1}{2}D\int_A[(\Delta w)^2-2(1-\nu)L(w,w)]\,\mathrm{d}A+\tfrac{1}{2}\frac{h}{E}\int_A[(\Delta\phi)^2-2(1+\nu)L(\phi,\phi)]\,\mathrm{d}A, \tag{10}$$

where $A = ab$ is the PCB area, m is its mass per unit area (with consideration of the masses of the SM components, assuming that these masses can be uniformly "spread" on the PCB surface), $D = Eh^3/12(1-\nu^2)$ is the PCB's flexural rigidity (we assume that the SM components are small and affect only the PCB mass, but not its rigidity), h is the board's thickness, and

$$\Delta = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$$

is the Laplace operator. The first term in (10) is due to bending, and the second term is due to the tensile membrane stresses.

After substituting (4) into formulas (9) and (10), we obtain:

$$T = \tfrac{1}{2}(m\dot{w}_c^2A-2M_0\dot{w}_c\dot{z}+M\dot{z}^2),\quad V = \tfrac{1}{2}M(\lambda^2z^2+\tfrac{1}{2}\alpha z^4), \tag{11}$$

where the following notation is used:

$$\left.\begin{aligned} M_0 &= m\int_A w_1\,\mathrm{d}A,\quad M = m\int_A w_1^2\,\mathrm{d}A,\\ \lambda^2 &= \frac{D}{M}\int_A[(\Delta w_1)^2-(1-\nu)L(w_1,w_1)]\,\mathrm{d}A,\\ \alpha &= \frac{2h}{EM}\int_A[(\Delta\phi_1)^2-2(1+\nu)L(\phi_1,\phi_1)]\,\mathrm{d}A\end{aligned}\right\}. \tag{12}$$

Introducing expressions (11) for the energies in the Lagrange equation [see, for instance, Pars (1965)]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{z}}+\frac{\partial V}{\partial z}=0,$$

we obtain the following nonlinear differential equation for the unknown principal coordinate $z(t)$:

$$\ddot{z}+\lambda^2 z+\alpha z^3 = q. \tag{13}$$

Here

$$q = c\ddot{w}_c \tag{14}$$

is the excitation force, and

$$c=\frac{M_0}{M}=\frac{\int_A w_1\,\mathrm{d}A}{\int_A w_1^2\,\mathrm{d}A} \tag{15}$$

is the factor considering the effect of the coordinate function on the magnitude of the excitation force. Equation (13) describes nonlinear forced vibrations of a one-degree-of-freedom system with the linear frequency λ and the parameter of nonlinearity α [see, for instance, Den Hartog (1956)].

With eqns (5) and (6), formulas (12) and (15) result, in the case of a simply-supported board, in the following expressions for the linear vibration frequency λ, parameter of nonlinearity α, and the excitation force factor c:

$$\left.\begin{aligned}\lambda &= \pi^2\frac{a^2+b^2}{a^2b^2}\sqrt{\frac{D}{m}}\\ \alpha &= \frac{3\pi^4}{4}\frac{D}{mh^2}\frac{(3-\nu^2)(a^4+b^4)+4\nu a^2b^2}{a^4b^4}\\ c &= \frac{16}{\pi^2}=1.621\end{aligned}\right\}. \tag{16}$$

For a square board ($a = b$) we have:

$$\lambda=\frac{2\pi^2}{a^2}\sqrt{\frac{D}{m}},\quad \alpha=\frac{\pi^4}{8}\frac{3-\nu}{1-\nu}\frac{Eh}{ma^4}. \tag{17}$$

In the case of a clamped board, introducing expressions (7) and (8) into formulas (12), we obtain:

$$\left.\begin{aligned}\lambda &= \frac{4\pi^2}{3a^2b^2}\sqrt{[3(a^4+b^4)+2a^2b^2]\frac{D}{m}},\\ \alpha &= \frac{\pi^4 Eh}{18ma^4b^4}\left[\frac{9(a^4+b^4+2\nu a^2b^2)}{4(1-\nu^2)}+\frac{17}{8}(a^4+b^4)+\frac{12a^4b^4}{(a^2+b^2)^2}+\frac{5a^4b^4}{(4a^2+b^2)^2}+\frac{5a^4b^4}{(a^2+4b^2)^2}\right],\\ c &= (\tfrac{4}{3})^2 = 1.778\end{aligned}\right\}. \tag{18}$$

For a square board ($a = b$)

$$\lambda = \frac{8\pi^2}{3a^2}\sqrt{\frac{2D}{m}}, \quad \alpha = \frac{\pi^4}{49}\frac{27-17\nu}{1-\nu}\frac{Eh}{ma^4}. \tag{19}$$

For an elongated board the exact coordinate function is as follows [see, for instance, Timoshenko and Young (1955)]:

$$w_1 = C\cosh\beta\frac{x}{a} + \cos\beta\frac{x}{a}, \tag{20}$$

where $C = 0$, $\beta = \pi$ in the case of a simply-supported board, and $C = 0.1329$, $\beta = 4.73$ in the case of a clamped board. This results in the formulas

$$\lambda = \frac{\pi^2}{a^2}\sqrt{\frac{D}{m}}, \quad \alpha = \frac{\pi^4}{4}\frac{1}{1-\nu^2}\frac{Eh}{ma^4}, \quad c = \frac{4}{\pi} = 1.273 \tag{21}$$

for a simply-supported board, and in the formulas

$$\lambda = \frac{10}{a^2}\sqrt{\frac{5D}{m}}, \quad \alpha = \frac{11.75}{1-\nu^2}\frac{Eh}{ma^4}, \quad c = 1.165 \tag{22}$$

for a clamped board.

Maximum deflection, velocity and acceleration

The maxima of the deflection, velocity and acceleration can be determined even without solving eqn (13). Indeed, with the constant external acceleration $\ddot{w}_c$, this equation can be written as

$$\frac{d}{dt}(\dot{z}^2 + \lambda^2 z^2 + \tfrac{1}{2}\alpha z^4 - 2qz) = 0,$$

or

$$\dot{z}^2 + \lambda^2 z^2 + \tfrac{1}{2}\alpha z^4 - 2qz = C.$$

If the initial displacement and velocity are zero, the constant of integration C is also zero, and therefore

$$\dot{z} = \sqrt{2qz - \lambda^2 z^2 - \tfrac{1}{2}\alpha z^4}. \tag{23}$$

This relationship (phase diagram) is plotted in Fig. 2.

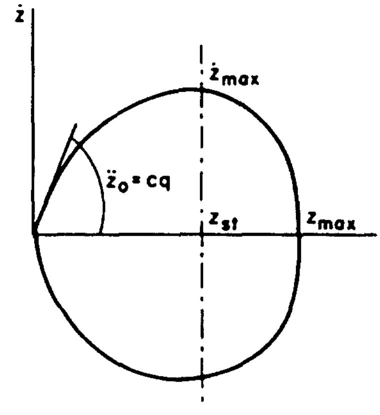

Fig. 2. Phase diagram for the principal coordinate.

The displacement z reaches its maximum value z_{max} at the end of the first quarter-period of vibrations, when the induced velocity $\dot{z}$ is zero. This results in the equation:

$$2q - \lambda^2 z_{max} - \tfrac{1}{2}\alpha z_{max}^3 = 0. \tag{24}$$

If the force q were applied statically, then, putting in eqn (13) $\ddot{z} = 0$, we obtain the following cubic equation for the static displacement z_{st}:

$$q - \lambda^2 z_{st} - \alpha z_{st}^3 = 0. \tag{25}$$

Since the maximum induced velocity $\dot{z}_{max}$ also takes place when the acceleration $\ddot{z}$ is zero, we conclude, that $\dot{z} = \dot{z}_{max}$, when $z = z_{st}$, so that

$$\dot{z}_{max} = \sqrt{2qz_{st} - \lambda^2 z_{st}^2 - \tfrac{1}{2}\alpha z_{st}^4}. \tag{26}$$

The cubic eqns (24) and (25) have the following solutions:

$$z_{max} = \eta_z z_{max}^0, \tag{27}$$

$$z_s = \eta_{st} z_{st}^0, \tag{28}$$

where

$$z_{max}^0 = \frac{2q}{\lambda^2} \tag{29}$$

is the maximum linear dynamic displacement,

$$z_{st}^0 = \frac{q}{\lambda^2} = \frac{z_{max}^0}{2} \tag{30}$$

is the maximum linear static displacement, and the factors

$$\eta_z = \frac{1}{2\sqrt[3]{\mu}}\left(\sqrt[3]{1+\sqrt{1+\frac{1}{27\mu}}} + \sqrt[3]{1-\sqrt{1+\frac{1}{27\mu}}}\right), \quad \mu = \frac{\alpha}{2}\left(\frac{z_{max}^0}{\lambda}\right)^2, \tag{31}$$

$$\eta_{st} = \frac{1}{\sqrt[3]{\mu}}\left(\sqrt[3]{1+\sqrt{1+\frac{8}{27\mu}}} + \sqrt[3]{1-\sqrt{1+\frac{8}{27\mu}}}\right) \tag{32}$$

consider the effect of the nonlinearity on the maximum dynamic and static displacements, respectively. The dynamic factor

$$K_d = \frac{z_{max}}{z_{st}} = \frac{\sqrt[3]{1+\sqrt{1+1/27\mu}} + \sqrt[3]{1-\sqrt{1+1/27\mu}}}{\sqrt[3]{1+\sqrt{1+8/27\mu}} + \sqrt[3]{1-\sqrt{1+8/27\mu}}} \tag{33}$$

changes from $K_d = 2$ in a linear system ($\mu = 0$) to $K_d = 1$ in a strongly nonlinear system

μ	0	0.2	0.5	1.0	2.0	3.0	4.0
η_z	1	0.7117	0.5897	0.5000	0.4176	0.3737	0.3447
η_{st}	1	0.9217	0.8477	0.7709	0.6823	0.6282	0.5898
K_d	2	1.5493	1.3913	1.2972	1.2240	1.1898	1.1689
$\eta_{\ddot z}$	1	1.263	1.458	1.636	1.821	1.927	2.000

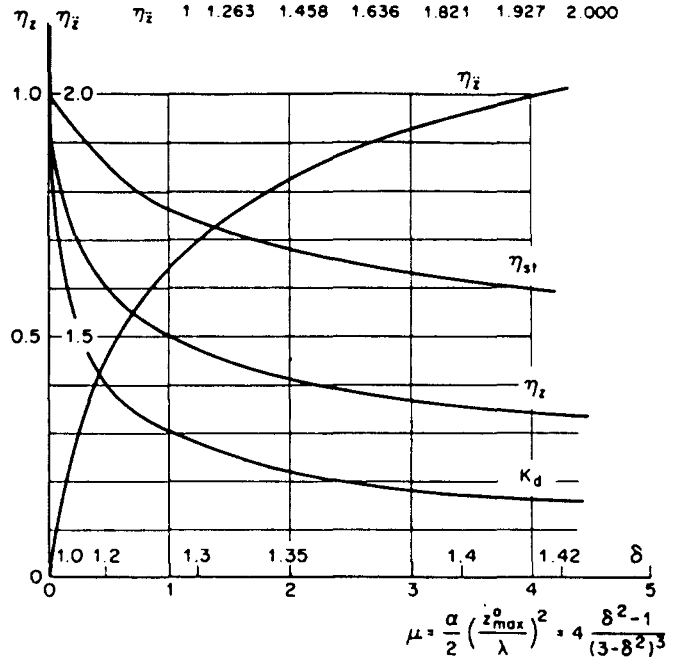

Fig. 3. Factors reflecting the effect of nonlinearity on the dynamic (η_z) and static (η_{st}) displacements, on the induced acceleration ($\eta_{\ddot z}$) and the dynamic factor (K_d), as functions of the dimensionless parameter μ of nonlinearity.

($\mu \to \infty$). The factors η_z, η_{st} and K_d are plotted in Fig. 3. This figure indicates that dynamic nonlinear effects are substantially greater than the static effects.

The initial elastic (induced) acceleration can be easily obtained from eqn (13) by putting the displacement z equal to zero:

$$\ddot z_0 = q = c\ddot w_c. \tag{34}$$

Thus, the factor c, given by formula (15), is, in effect, the ratio of the maximum initial elastic acceleration to the acceleration of the contour. The distribution of the total (absolute) acceleration over the surface of the PCB can be obtained on the basis of the first formula in (4) and is as follows:

$$\ddot w_1 = \ddot w_c - w_1(x, y)\ddot z_0 = -[cw_1(x, y) - 1]\ddot w_c. \tag{35}$$

It is evident from this formula that the initial acceleration $\ddot w_1$ is the maximum on the contour (where $w_1 = 0$), and is the minimum in the center of the board (where $w_1 = 1$):

$$\ddot w_{1,min} = -(c-1)\ddot w_c. \tag{36}$$

In the case of a simply-supported board of finite aspect ratio, using (5), we have:

$$\ddot w_1(x, y) = -\left(\frac{16}{\pi^2}\cos\frac{\pi x}{a}\cos\frac{\pi y}{b} - 1\right)\ddot w_c.$$

This formula indicates that the initial accelerations are negative within the rectangular

$$\frac{x}{a} = \frac{y}{b} = \frac{1}{\pi} \arccos \frac{\pi^2}{16} = 0.2884.$$

The minimum acceleration in the center of the board, in accordance with (36), is $\ddot{w}_{i,\min} = -0.621\ddot{w}_c$. In the case of a clamped board, whose coordinate function is expressed by (7), we obtain that the initial negative accelerations occur inside the rectangular $x/a = y/b = 0.3098$ and that the minimum acceleration is $\ddot{w}_{i,\min} = -0.778\ddot{w}_c$. Thus, negative initial accelerations occupy rather large PCB areas and their absolute maxima are quite comparable with the magnitude of the external acceleration. This is important to have in mind, particularly when evaluating the strength of interconnections of the devices mounted in the inner portion of the PCB on its "front" side, i.e. on the side of the direction of motion. At the first moments of loading such interconnections may experience rather high tensile stresses. These have a potential to be more dangerous than even larger compressive stresses that will occur later, at the end of the first quarter-period of vibrations.

In elongated boards the region of negative initial accelerations is $-0.212a \leqslant x \leqslant 0.212a$ for a simply-supported board, and $-0.169a \leqslant x \leqslant 0.169a$ for a clamped board. The accelerations in the center of the board are $\ddot{w}_{i,\min} = -0.273\ddot{w}_c$ and $\ddot{w}_{i,\min} = -0.297\ddot{w}_c$, respectively.

As one can see from eqns (13) and (25), the induced acceleration of the board becomes zero when $z = z_{st}$. At this moment in time all the points of the board have the same acceleration as the support contour.

At the end of the first quarter-period of vibrations, when the board reaches its maximum deflection $z = z_{\max}$, its elastic acceleration, as follows from (13), is

$$\ddot{z}_{\max} = q - \lambda^2 z_{\max} - \alpha z_{\max}^3,$$

or, considering (24),

$$\ddot{z}_{\max} = \lambda^2 z_{\max} - 3q. \tag{37}$$

From (29) and (37) we find that the induced linear acceleration is

$$\ddot{z}^0_{\max} = -q, \tag{38}$$

i.e. equal in magnitude and opposite in sign to the initial (linear or nonlinear) acceleration, expressed by formula (34). The formulas (37) and (38) indicate that the factor

$$\eta_{\ddot{z}} = \frac{\ddot{z}_{\max}}{\ddot{z}^0_{\max}} = 3 - \frac{\lambda^2 z_{\max}}{q} = 3 - 2\eta_z \tag{39}$$

accounts for the effect of the nonlinearity on the maximum induced acceleration. This factor is plotted in Fig. 3. In strongly nonlinear systems ($\mu \to \infty$) the factor η_z, reflecting the effect of the nonlinearity on the maximum deflection, is very small, and therefore in such an extreme case the nonlinear induced acceleration can exceed the linear acceleration by a factor of 3.

Consider a new dimensionless parameter δ, which will be used hereafter, so that

$$\mu = \frac{\alpha}{2}\left(\frac{\ddot{z}^0_{\max}}{\lambda}\right)^2 = 4\frac{\delta^2 - 1}{(3-\delta^2)^3}. \tag{40}$$

Let us show that $\eta_{\ddot{z}} = \delta^2$. Indeed, in this case eqn (39) yields: $\eta_z = (3-\delta^2)/2$, and formula (40) results in the following equation for the factor η_z: $1 - \eta_z - \mu\eta_z^3 = 0$. Since $\eta_z = \lambda^2 z_{\max}/2q$, this cubic equation yields: $2q - \lambda^2 z_{\max} - \frac{1}{2}\alpha z_{\max}^3 = 0$, i.e. leads to the previously-obtained eqn (24). The calculated values of δ are shown in Fig. 3 along with the values of μ. When μ changes from zero to infinity, the δ value changes from 1 to $\sqrt{3}$.

The absolute (total) accelerations of the board at the moment of time equal to the quarter-period of vibrations are

$$\ddot{w}_{max} = \ddot{w}_c - w_1(x,y)\ddot{z}_{max} = [1 + c\eta_z w_1(x,y)]\ddot{w}_c. \tag{41}$$

It is evident from this formula that all the points of the board have at this moment in time the same direction of their maximum accelerations, as the support contour. At the center of the board they are by a factor of

$$\eta_0 = 1 + c\eta_z \tag{42}$$

larger than on the support contour. In a linear system this factor is $\eta_0 = 1 + c = 2.621$ for a small aspect ratio simply-supported board, $\eta_0 = 2.273$ for an elongated simply-supported board, $\eta_0 = 2.778$ for a small aspect ratio clamped board, and $\eta_0 = 2.165$ for an elongated clamped board. In strongly nonlinear systems, with the factor of the elastic acceleration η_z approaching 3, the factor η_0 of the total acceleration at the center of the board reaches $\eta_0 = 5.863$ in the case of a small aspect ratio simply-supported board, $\eta_0 = 4.819$ in the case of an elongated simply-supported board, $\eta_0 = 6.334$ in the case of a small aspect ratio clamped board, and $\eta_0 = 4.495$ in the case of an elongated clamped board. Thus, nonlinearity can result in a significant increase in the total acceleration of the PCB. Obviously, at this moment in time the interconnections in the devices mounted on the "front" side of the board experience compressive stresses, while the interconnections on its "back" side are subjected to tension. It should be pointed out that because of the structural damping the elastic vibrations fade in the course of time, and therefore at the moments of time sufficiently remote from the moment of loading, the PCB accelerations are simply equal to the acceleration $\ddot{w}_c$ of the contour.

Solution to the equation of motion

The maxima of the PCB deflection, velocity and acceleration were determined in the previous section on the basis of more or less elementary considerations, without solving the equation of motion (13). This solution can be written, using (23), in the form:

$$t = \int_0^z \frac{dz}{\sqrt{2qz - \lambda^2 z^2 - \frac{1}{2}\chi z^4}} = \frac{u}{\sigma}, \tag{43}$$

where

$$u = F(\theta, k) = \int_0^\theta \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \tag{44}$$

is the elliptic integral of the first kind [see, for instance, Bateman and Erdelyi (1955); Abramowitz and Stegun (1964); Sneddon (1980)], k is the modulus of the elliptic function, $\theta = \text{cn}\, u$ is the amplitude of this function, and σ is the frequency parameter.

In order to express the parameters k and σ through the parameters of the dynamic system (13), we seek the inversion of the above integrals in the form:

$$z = z_{max} \frac{1 - \text{cn}\, u}{\delta + 1 + (\delta - 1)\text{cn}\, u}. \tag{45}$$

Here z_{max} is the maximum displacement, cn u is the elliptic cosine, and δ is a so-far unknown parameter. Using the rules of differentiation of the elliptic functions, we obtain:

$$\dot{z} = 2\delta\sigma z_{max}\frac{\operatorname{sn} u \operatorname{dn} u}{[\delta+1+(\delta-1)\operatorname{cn} u]^2}, \tag{46}$$

$$\ddot{z} = 2\delta\sigma^2 z_{max}\frac{(\delta+1)(1-2k^2\operatorname{sn}^2 u)\operatorname{cn} u+(\delta-1)[1+(1-2k^2)\operatorname{sn}^2 u]}{[(\delta+1)+(\delta-1)\operatorname{cn} u]^3}, \tag{47}$$

where sn u is the elliptic sine, and dn $u = \sqrt{1-k^2\operatorname{sn}^2 u}$ is the function of delta-amplitude. After substituting (45) and (47) into (13), we conclude that the maximum displacement z_{max} is expressed by eqn (24), and the parameters k and σ are as follows:

$$k = \sqrt{\frac{(\delta-1)(3-\delta)}{8\delta}}, \quad \sigma = \lambda\sqrt{\frac{2\delta}{3-\delta^2}}, \tag{48}$$

where the parameter

$$\delta = \sqrt{1+\frac{\alpha}{2q}z_{max}^3} = \sqrt{3-\frac{\lambda^2}{2}z_{max}} = \sqrt{3-2\eta_z} \tag{49}$$

is related to the dimensionless parameter of nonlinearity μ, introduced earlier, by eqn (40). In the linear case ($\alpha = 0$), $\delta = 1$, $k = 0$ and $\sigma = \lambda$. In a strongly-nonlinear case ($\alpha \to \infty$), $z_{max} = 0$, $\delta = \sqrt{3}$, $k = \frac{1}{2}\sqrt{2-\sqrt{3}} = 0.259$, and $\sigma \to \infty$. The elliptic functions, entering the formulas (45), (46) and (47), can be computed using recommendations and formulas contained in Lance (1960).

All the relationships of the previous section can be determined, of course, on the basis of the solutions of this section. Let us show, for instance, how relationship (37) can be obtained from (45) and (47). From (45) we find that cn $u = 1$ when $z = z_{max}$. Since in this case sn $u = 0$, eqn (47) yields: $\ddot{z} = \frac{1}{2}\delta\sigma^2 z_{max}$. Then from (48) and (49) we have: $\delta\sigma^2 = -(2/z_{max})(\lambda^2 z_{max}-3q)$. This leads to (37).

The formula for the amplitude θ of the elliptic function can be obtained from (45), assuming cn $u = \cos\theta$. This results in the equation

$$\theta = \operatorname{arccotan}\sqrt{\frac{1}{\delta}\left(\frac{z_{max}}{z}-1\right)}.$$

The amplitude θ reaches its maximum value $\theta_{max} = \pi/2$ when the displacement z reaches z_{max}. In this case the integral (44) becomes a complete elliptic integral of the first kind:

$$K(k) = F\left(\frac{\pi}{2}, k\right) = \int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}.$$

Since the time required for the angle θ to change form zero to $\pi/2$ is equal to the quarter of the period of vibrations, we conclude, on the basis of solution (43), that this period is $4K(k)/\sigma$. Therefore, the vibration frequency is

$$\omega = \frac{\pi\sigma}{2K(k)}. \tag{50}$$

In the linear case, $k = 0$, $K(0) = \pi/2$, $\delta = 1$ and $\omega = \sigma = \lambda$.

As follows from the obtained results, the effect of the nonlinearity on the maximum induced (relative) displacement, velocity, acceleration and frequency can be characterized by the following factors:

$$\eta_z = \frac{3-\delta^2}{2}, \quad \eta_z = \sqrt{\tfrac{1}{2}\eta_{st}(3-\eta_{st})}, \quad \eta_z = \delta^2, \quad \eta_\omega = \frac{\pi}{2K(k)}\frac{\sigma}{\lambda} = \frac{\pi}{2K(k)}\sqrt{\frac{2\delta}{3-\delta^2}}.$$

Stress

The stresses arising in the PCB can be computed on the basis of the following formulas for the rectangular plates [see, for instance, Timoshenko and Woinowski-Krieger (1969)]:

$$\left.\begin{aligned} \sigma_x &= \sigma_x^0 - \frac{6D}{h^2}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) \\ \sigma_y &= \sigma_y^0 - \frac{6D}{h^2}\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) \\ \tau_{xy} &= \tau_{xy}^0 - \frac{6D}{h^2}(1-\nu)\frac{\partial^2 w}{\partial x\,\partial y} \end{aligned}\right\}, \tag{51}$$

where the first terms are the membrane stresses, expressed by the formulas (1), and the second terms are due to bending.

With eqns (5) and (6) for the coordinate function and the static stress function, we find, using (51), that the maximum normal stresses in a simply-supported board of finite aspect ratio occur in its center ($x = y = 0$) and are as follows:

$$\left.\begin{aligned} \sigma_x &= \frac{\pi^2 Eh^2}{2a^2b^2}\frac{\nu a^2+b^2}{1-\nu^2}\zeta\left[1+\frac{\nu a^2+(2-\nu^2)b^2}{4(\nu a^2+b^2)}\zeta\right] \\ \sigma_y &= \frac{\pi^2 Eh^2}{2a^2b^2}\frac{a^2+\nu b^2}{1-\nu^2}\zeta\left[1+\frac{(2-\nu^2)a^2+\nu b^2}{4(a^2+\nu b^2)}\zeta\right] \end{aligned}\right\}, \tag{52}$$

where $\zeta = z_{max}/h$ is the dimensionless maximum displacement. The maximum shearing stress occurs along the rectangular $x = \pm a/2$, $y = \pm b/2$ and is

$$\tau_{xy} = \frac{\pi^2 Gh^2}{ab}\zeta, \tag{53}$$

where $G = E/2(1+\nu)$ is the shear modulus of the PCB material. In a special case of a square board ($a = b$) we have:

$$\sigma_x = \sigma_y = \frac{\pi^2 E}{2(1-\nu)}\left(\frac{h}{a}\right)^2\zeta\left(1+\frac{2-\nu}{4}\zeta\right), \quad \tau_{xy} = \pi^2 G\left(\frac{h}{a}\right)^2\zeta. \tag{54}$$

In the case of an elongated board ($b \to \infty$) the shearing stress is zero, and the maximum normal stress, calculated on the basis of eqn (20), is

$$\sigma_x = \frac{\pi^2 E}{2(1-\nu^2)}\left(\frac{h}{a}\right)^2\zeta\left(1+\frac{\zeta}{2}\right). \tag{55}$$

Note that the first formula in (52), when $b \to \infty$, yields:

$$\sigma_x = \frac{\pi^2 E}{2(1-\nu^2)}\left(\frac{h}{a}\right)^2\zeta\left(1+\frac{2-\nu^2}{4}\zeta\right). \tag{56}$$

Hence, the results obtained on the basis of the formulas (55) and (56) are quite close.

Equations (7) and (8) result in the following normal stresses in the center of a clamped board:

$$\sigma_x = \frac{\pi^2 E h^2}{a^2 b^2}\frac{\nu a^2 + b^2}{1-\nu^2}\zeta\left\{1 + \frac{1-\nu^2}{32}\frac{b^2}{\nu a^2 + b^2}\zeta\left[5 + \frac{3}{1-\nu^2}\left(1+\nu\frac{a^2}{b^2}\right) + \frac{8a^2b^2}{(a^2+b^2)^2} + \frac{16a^2b^2}{(4a^2+b^2)^2} + \frac{4a^2b^2}{(a^2+4b^2)^2}\right]\right\}$$

$$\sigma_y = \frac{\pi^2 E h^2}{a^2 b^2}\frac{a^2 + \nu b^2}{1-\nu^2}\zeta\left\{1 + \frac{1-\nu^2}{32}\frac{a^2}{a^2 + \nu b^2}\zeta\left[5 + \frac{3}{1-\nu^2}\left(1+\nu\frac{b^2}{a^2}\right) + \frac{8a^2b^2}{(a^2+b^2)^2} + \frac{4a^2b^2}{(4a^2+b^2)^2} + \frac{16a^2b^2}{(a^2+4b^2)^2}\right]\right\}. \quad (57)$$

The maximum shearing stress in a clamped board of finite aspect ratio occurs along the perimeter of a rectangle $x = \pm a/4$, $y = \pm b/4$:

$$\tau_{xy} = \frac{\pi^2 G h^2}{ab}\zeta\left[1 + \frac{1+\nu}{2}\frac{a^2b^2}{(a^2+b^2)^2}\zeta\right]. \quad (58)$$

In the case of a square clamped board ($a = b$)

$$\left.\begin{aligned}\sigma_x = \sigma_y &= \frac{\pi^2 E}{1-\nu}\left(\frac{h}{a}\right)^2\zeta\left[1 + \frac{3(18-13\nu)}{160}\zeta\right]\\ \tau_{xy} &= \pi^2 G\left(\frac{h}{a}\right)^2\zeta\left(1+\frac{1+\nu}{8}\zeta\right)\end{aligned}\right\}. \quad (59)$$

The maximum normal stress in an elongated clamped board is

$$\sigma_x \cong 16\frac{E}{1-\nu^2}\left(\frac{h}{a}\right)^2\zeta(1+0.193\zeta). \quad (60)$$

NUMERICAL EXAMPLE

Let an ASTM/NEMA Class G-10 fiber-glass PCB simply-supported on its contour be subjected to a constant suddenly-applied acceleration $\ddot{w}_c = 25$ g applied to the contour. Let the weight of all the SM devices be 20% of the board's weight, and let there be a certain flexibility in the spot where the given SM device can be installed on the board. In addition, let the given device be able to withstand accelerations, not exceeding, say, 100 g. Our purpose is to determine the PCB areas where the device can be safely installed, so that its strength and the reliable operation are not compromised. We use the following input data: density of the PCB material $\rho = 0.065$ lb in^{-3} = 1.8 g cm^{-3} = 17655 N m^{-3}, Young's modulus $E = 2.45 \times 10^6$ psi = 17.2×10^4 kgf cm^{-2} = 16.9 GPa, Poisson's ratio $\nu = 0.3$, ultimate stress in tension $\sigma_u = 36000$ psi $\cong$ 2500 kgf cm^{-2} = 0.248 GPa, thickness $h = 0.0625$ in. = 0.159 cm, and the in-plane dimensions are $a = 8$ in. = 20.3 cm and $b = 12$ in. = 30.5 cm (the aspect ratio is $b/a = 1.5$).

Using formulas (16), we find that the linear frequency of free vibrations of the board is $\lambda = 463.5\,\mathrm{s}^{-1}$, and the parameter of nonlinearity is $\alpha = 12.3 \times 10^6\,\mathrm{cm}^{-2}\,\mathrm{s}^{-2}$. The excitation force acting on the PCB contour, in accordance with formula (14), is $q = 1.621 \times 25$ g = 39715 cm s^{-2}. The maximum linear dynamic displacement, determined by (29), is $z^0_{max} = 2q/\lambda^2 = 0.370$ cm. The dimensionless parameter of nonlinearity, given by (31), is

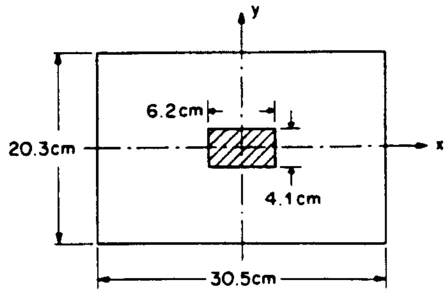

Fig. 4. The component in question cannot be placed, for safe operation, within the shaded area.

$\mu = \frac{1}{2}\chi(z^0_{max}/\lambda)^2 = 3.905$. Then we find that the factor considering the effect of the nonlinearity on the maximum deflection is $\eta_z = 0.347$, so that this deflection is $z_{max} = \eta_z z^0_{max} = 0.128$ cm. The factor accounting for the effect of the nonlinearity on the maximum elastic (induced) acceleration, in accordance with formula (39), is $\eta_{\ddot{z}} = 3-2\eta_z = 2.306$. Then the distribution of the total (absolute) maximum accelerations over the board's surface, predicted by (41), is

$$\ddot{w}_{max} = 25\,g\left(1+3.738\cos\frac{\pi x}{a}\cos\frac{\pi y}{b}\right).$$

The condition $\ddot{w}_{max} \leqslant 100$ g results in the following equation for the rectangle restricting the region where the device can be safely mounted (Fig. 4):

$$\cos\frac{\pi x}{a}\cos\frac{\pi y}{b} = 0.803.$$

Thus, the device should be placed for safe operation outside the region $x/a = y/b = \pm 0.203$.

The maximum stresses, calculated in accordance with eqns (52) and (53), are $\sigma_x = 71.1$ kgf cm^{-2} = 6.97 MPa, $\sigma_y = 44.9$ kgf cm^{-2} = 4.40 MPa and $\tau_{xy} = 21.5$ kgf cm^{-2} = 2.11 MPa. These values are very small compared to the ultimate stress $\sigma_u = 2500$ kgf cm^{-2} = 245.2 MPa.

The maximum acceleration in the center of the board is $\ddot{w}_{max} = 118.45$ g. Note that the linear approach would lead to the following maximum acceleration in the PCB center: $\ddot{w}_{max} = (1+c)\ddot{w}_c = 2.621\ddot{w}_c = 65.5$ g, and would result in an erroneous conclusion that the device could be safely mounted anywhere on the board.

CONCLUSION

A simple and easy-to-use analytical model has been developed for the prediction of the maximum deflections, accelerations and stresses arising in a flexible plate (printed circuit board) due to a constant acceleration applied to its support contour. We showed that it is important to account for the nonlinearity of the plate (PCB) vibrations. If the applied acceleration and the induced deflections of the board are large, the nonlinear effects lead to significantly higher accelerations than the linear approach. Although this is true, generally speaking, also for the stresses, in the executed example these stresses turned out to be quite small. The results obtained can be used to evaluate the accelerations experienced by electronic components and devices, surface-mounted on flexible printed circuit boards. These results can also be of help when choosing the appropriate type, dimensions and support conditions of the PCB, as well as the most feasible layout of the components and devices.

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